Explorations 01 - Problem 9

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The problem reads: "Graph the following:"

$$x^2 + y^2 = 1$$
$$x^3 + y^3 = 1$$
$$x^4 + y^4 = 1$$
$$x^5 + y^5 = 1$$

and then asks, "What do you expect for the graph of:

$$x^{24} + y^{24} = 1$$
$$x^{25} + y^{25} = 1$$

First, let's look at the graphs:

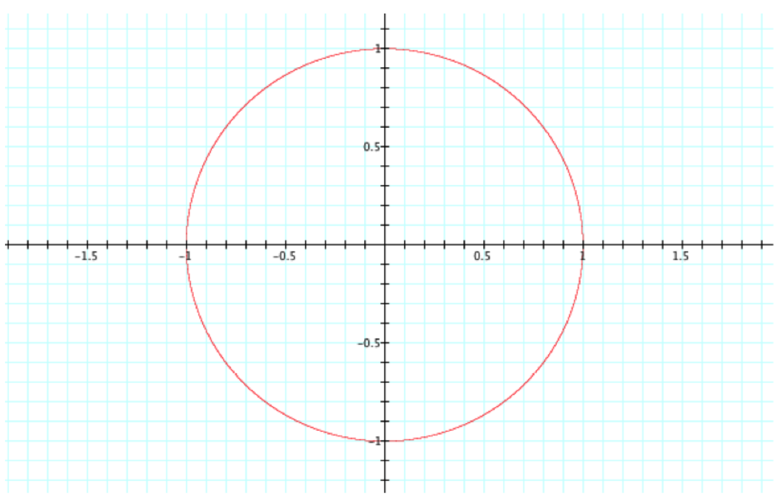

Figure 1: $x^2 + y^2 = 1$

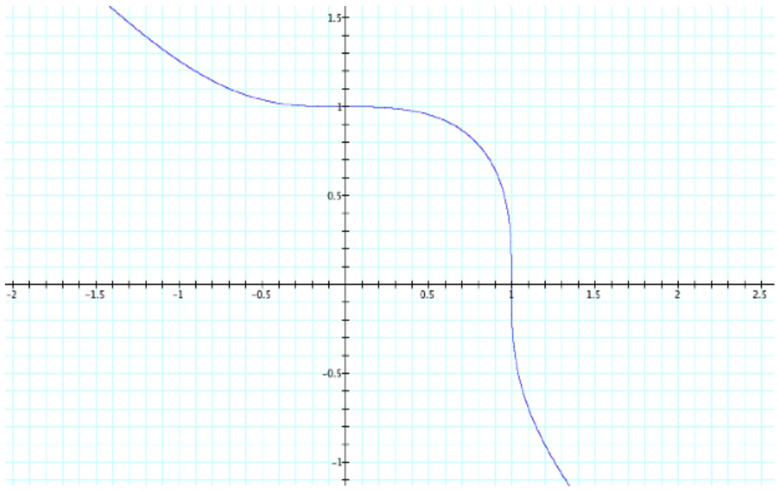

Figure 2: $x^3 + y^3 = 1$

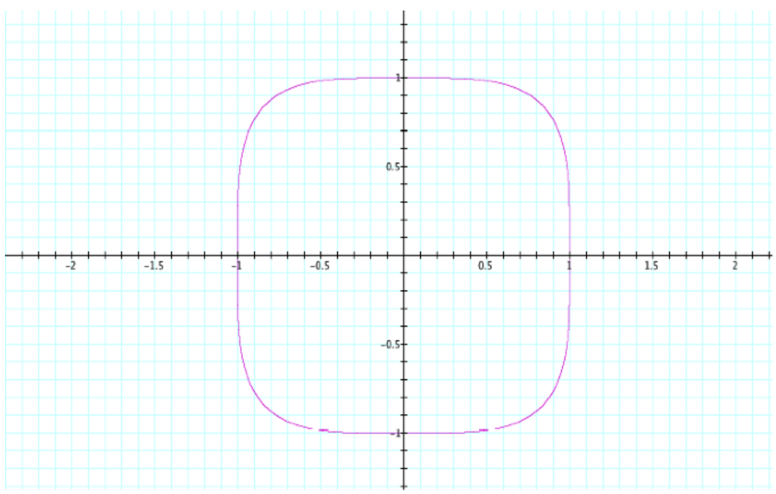

Figure 3: $x^4 + y^4 = 1$

Already, we notice a pattern. The first and third graphs - namely, $x^2 + y^2 = 1$ and $x^4 + y^4 = 1$ - have points (x, y) in all four quadrants. Indeed, there are (x, y) with $x, y < 0$ that satisfy each equation. On the other hand, the graph of $x^3 + y^3 = 1$ has values in only quadrants 1, 2, & 4. (★)This pattern is represented by the fact that $r^n > 0$ for all positive even n ($n \in 2\mathbb{N}$, or n of the form $n = 2k$, $k \in \mathbb{N}$), whereas $r^n < 0$ for all odd n when $r < 0$. Therefore, $x^n + y^n < 0 < 1 \ \forall x, y < 0$ when n is odd. We can thus anticipate that, since 5 is odd, the graph of $x^5 + y^5 = 1$ will have a similar shape as Figure 2.

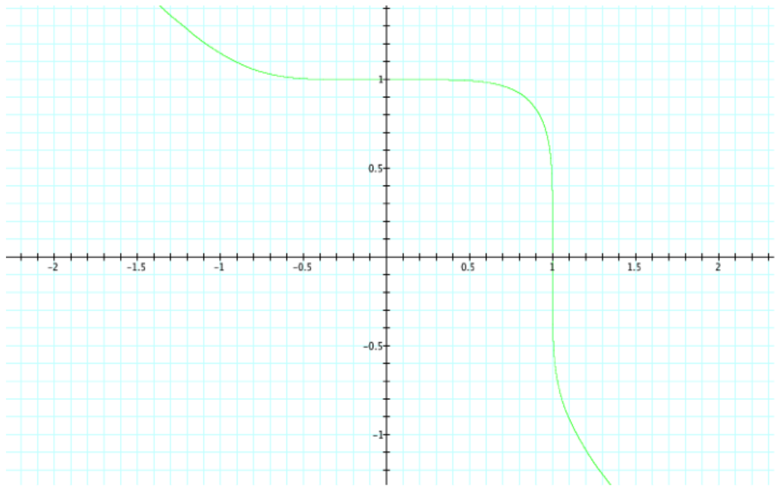

Figure 4: $x^5 + y^5 = 1$

Now, let us consider more detailed similarities and differences between the graphs. Figures 1 and 3 have convex, compact shapes contained in the Cartesian product $[-1, 1]$x$[-1, 1]$. That the figures are smooth (& continuous) is clear from the fact that

$$x^n + y^n = 1 \Rightarrow y = \sqrt[n]{1 - x^n}$$

is differentiable on $(-1, 1)$ and also defined at $x = 1$ and $x = -1$. From this same representation, it is clear that, for n even,

$$\sqrt[n]{1 - x^n} \leq \sqrt[n]{1} = 1$$

which determines the range of the function.

Now, the two figures are different in that Figure 3, or $x^4 + y^4 = 1$ more resembles a square than the well-known unit circle that $x^2 + y^2 = 1$ creates. In other words, the figure is more *rigid*. Rather than looking at the derivative, we will use a more intuitive approach to understand the mathematics behind the differences between Figures 1 and 3. We recognize that (for even n) as n increases, it becomes increasingly restrictive for the function $x^n + y^n$ to equal 1 when x^n is very close to 1. Observe Figures 1 and 3 on the same graph:

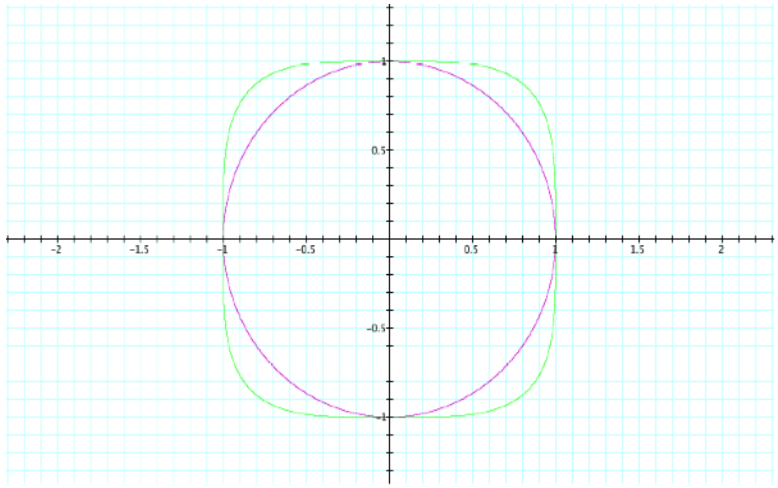

Figure 5: $x^2 + y^2 = 1$ in purple, $x^4 + y^4 = 1$ in green

For any x_0 near 1 but not equal to 1 such that $x_0^2 + y_1^2 = 1$ and $x_0^4 + y_2^4 = 1$, we have

$$x_0^4 < x_0^2 \Rightarrow y_2^4 > y_1^2 \Rightarrow y_1 < y_2^2 < y_2$$

for y_1, y_2 positive, $y_2 \in (0, 1)$. (the logic is very similar for $y_1, y_2 < 0$).
Therefore, we have (x_0, y_1) in Figure 1 closer to the x-axis on the line $x = x_0$ than (x_0, y_2) in Figure 3.
As this line of thinking applies to all $x \in (0, 1), y \in (0, 1)$ and can similarly be applied to all other $(x, y) \in [-1, 1]$ x $[-1, 1]$, we can now understand the relative rigidity of Figure 3 compared to Figure 1. Further, we can make an educated guess that, since the restriction that $x^n + y^n = 1$ puts onto y is even stronger for larger $n \in 2\mathbb{N}$, the graph of $x^{24} + y^{24} = 1$ will be even more rigid - might it resemble a square?

Let us now look at Figures 2 and 4, the graphs of $x^3 + y^3 = 1$ and $x^5 + y^5 = 1$. Both figures resemble the line $y = -x$ on the interval $(-\infty, -1) \cup (1, \infty)$, as shown below. Figures 2 and 4 have the same major disparity that Figures 1 and 3 did; the portion of $x^5 + y^5 = 1$ with $x \in (-1, 1)$ more closely resembles two sides of a square than its counterpart with lower exponents. Similarly, the graph of $x^5 + y^5 = 1$ "hugs" the graph of $y = -x$ much closer much sooner than Figure 3.

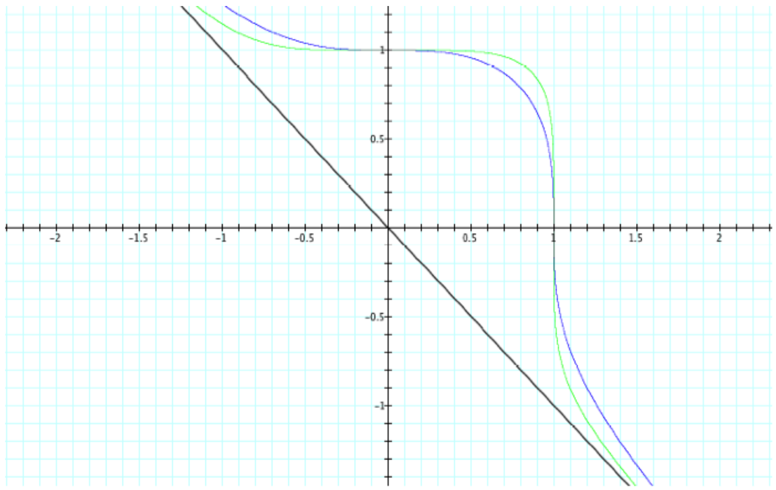

Figure 6: $x^3 + y^3 = 1$ in blue, $x^5 + y^5 = 1$ in green, $y = -x$ in black

By using arguments similar to those used to explain the shape of the graphs of Figures 1 and 3, we would be able to determine *why* the graph of $x^5 + y^5 = 1$ is more rigid than that of $x^3 + y^3 = 1$. What we must question now is
1) how and why the graphs of $x^n + y^n = 1$ differ for n even and odd,
2) what, if the graphs become more "rigid" as n increases, the limit of $x^n + y^n = 1$ is as $n \to \infty$ (and, with it, what the final rigid shape of each function will be)
The first set of questions is a natural result of the features of exponents discussed in ★. Whereas the graphs of the functions in question with n even are very compact (since $x^2 + y^2 = 1 \Rightarrow x, y \in [-1, 1]$), the graphs of such functions with n odd have no such restriction. If we consider the representation of $x^n + y^n = 1$ as

$$y^n = 1 - x^n$$

then it is clear that for large n odd, and $|x| > 1$, the exponential terms will dominate the 1 and the representation will approximate

$$y^n = -x^n$$

$$\Rightarrow y = -x$$

It is precisely the fact that x^n and y^n need not have the same sign that so increases the variability (i.e. larger domain and range) of $x^n + y^n = 1$ when n is odd.
Question(s) 1 is therefore answered; now for the final "form" of each group of functions. It seems from Figures 1 and 3 that even-degreed functions will approach a unit square. Indeed, if we consider the fact that,

$$\lim_{2k \to \infty} x^k = 0$$

for all $x \in (-1,1)$, we can determine that as $k \to \infty$, the only points in $[-1,1]$ x $[-1,1]$ that satisfy $x^{2k} + y^{2k} = 1$ are $\{(x, \pm 1) \mid x \in (-1,1)\}$ and $\{(1, \pm y) \mid y \in (-1,1)\}$ - exactly the unit square minus its corners.
For n odd, the behavior of $x^n + y^n = 1$ is nearly identical to the same function with $n+1$ or $n-1$ as the exponent of x and y. When n is odd, large, and $|x| > 1$, however, we should get almost exactly the line $y = -x$.

Given all of the above, we predict that $x^{24} + y^{24} = 1$ will look to the naked eye just like a square (with very slightly curved corners).

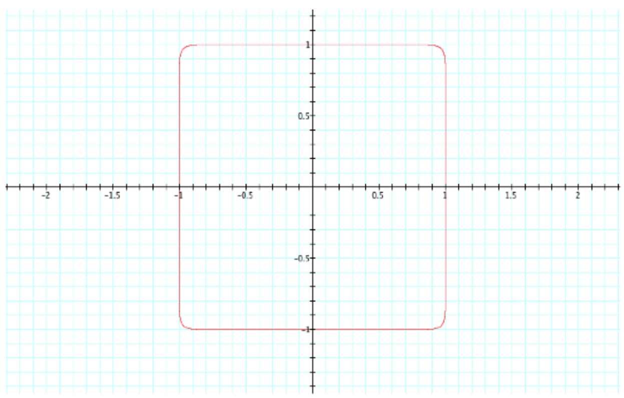

Figure 7: $x^{24} + y^{24} = 1$

Similarly, $x^{25} + y^{25} = 1$ should resemble the top right half of the unit square for $x \in [-1,1]$ and the line $y = -x$ for $x \in \mathbb{R} \setminus [-1,1]$.

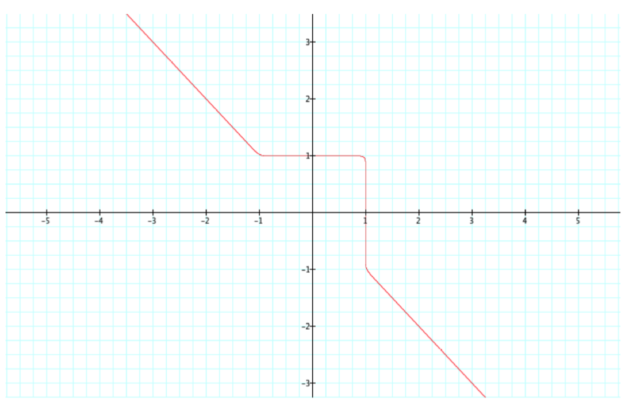

Figure 8: $x^{25} + y^{25} = 1$